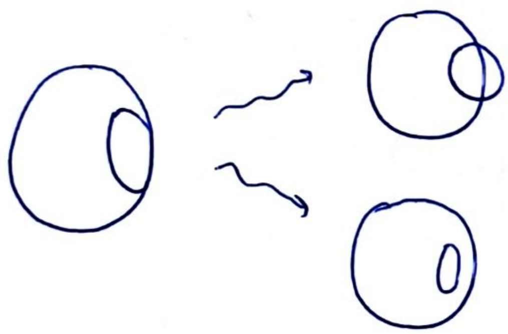
Math 132: Differential Topology

§ mod 2 intersection theory

Consider two closed submanifolds $M_1, M_2 \subset N$ of complimentary dimension (i.e. $\dim M_1 + \dim M_2 = \dim N$), where at least one of M_1 or M_2 is compact.

When they intersect transversely, $M_1 \cap M_2$ must be a compact 0-manifold (i.e. a finite set of points), so we can count the "intersection number" $\#(M_1 \cap M_2)$.

Even if their intersection is not transverse, we know we can homotope them to make it transverse.



Depending on the choice of homotopy, the number of intersections might change, but we'll show that mod 2 intersection number is well-defined.

Def Suppose M_1 is a compact manifold, $f: M_1 \to N$ is a smooth map, $M_2 \subset N$ is a closed submanifold with $\dim M_1 + \dim M_2 = \dim N$.

Choose any map g homotopic to f and transversal to M_2.

Define the mod 2 intersection number of f with M_2, $I_2(f, M_2)$, to be

$$\# g^{-1}(M_2) \bmod 2 \quad \in \mathbb{Z}/2\mathbb{Z}.$$

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<u>Thm</u> If $g_0, g_1 : M_1 \to N$ are homotopic and both transversal to $M_2 \subset N$, then $\# g_0^{-1}(M_2) = \# g_1^{-1}(M_2) \bmod 2$.

In particular, <u>the mod 2 intersection number is well-defined and invariant under homotopy</u>.

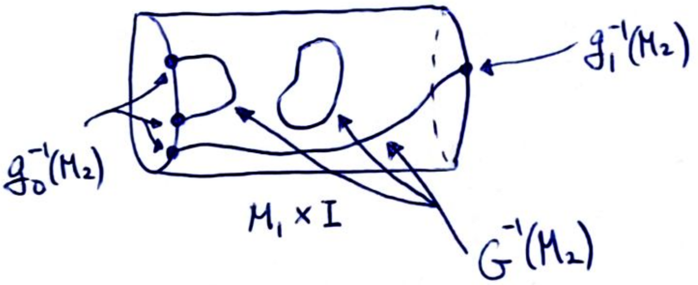
proof) Let $G : M_1 \times I \to N$ be a homotopy of g_0 and g_1.

By the relative transversality homotopy theorem, we may assume $G \pitchfork M_2$.

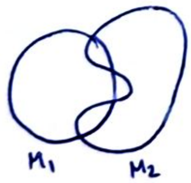
Then $G^{-1}(M_2)$ is a 1-dim'l submanifold of $M_1 \times I$ with boundary

$$\partial G^{-1}(M_2) = G^{-1}(M_2) \cap \partial(M_1 \times I) = g_0^{-1}(M_2) \times \{0\} \cup g_1^{-1}(M_2) \times \{1\}.$$

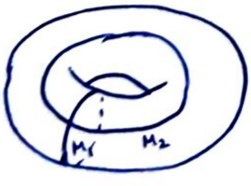
$\blacksquare$



<u>Ex</u>

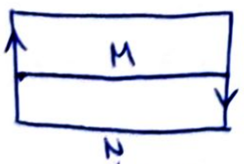


$\rightsquigarrow I_{\mathbb{Z}_2}(M_1, M_2) = 0$



$\rightsquigarrow I_{\mathbb{Z}_2}(M_1, M_2) = 1$

mod 2 self-intersection number!



Möbius band

$\rightsquigarrow I_{\mathbb{Z}_2}(M, M) = 1$

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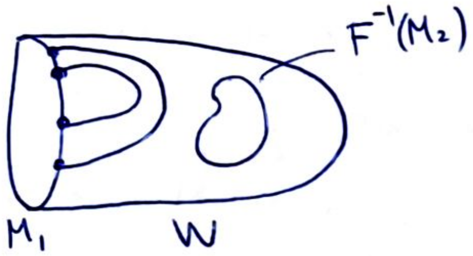
Thm (boundary theorem)

Suppose that M_1 is the boundary of some compact manifold W and $f: M_1 \to N$ is a smooth map. If f can be extended to all of W, then $I_{\mathbb{Z}/2}(f, M_2) = 0$ for any closed submanifold $M_2 \subset N$ of complementary dim.

proof) Let $F: W \to N$ extend f, which we may assume to be transversal to M_2, by the transversality homotopy thm. Then $F^{-1}(M_2)$ is a compact 1-manifold with boundary $\partial F^{-1}(M_2) = f^{-1}(M_2)$, so $\# f^{-1}(M_2)$ must be even. ■



Def If $f: M \to N$ is a smooth map of a compact manifold M into a connected manifold N of the same dimension, define the mod 2 degree of f $\deg_{\mathbb{Z}/2}(f)$, to be $I_{\mathbb{Z}/2}(f, \{y\})$, for any point $y \in N$.

Thm Mod 2 degree is well-defined.

proof) For any $y \in N$, we can homotope f to make it transversal to $\{y\}$.
From lecture 4, we know $y \mapsto I_{\mathbb{Z}/2}(f, \{y\})$ is locally constant.
Since N is connected, it must be globally constant. ■

Since $\deg_{\mathbb{Z}/2}$ is defined as an intersection number, we immediately obtain:

<u>Thm</u> The mod 2 degree is invariant under homotopy.

<u>Thm</u> If $M = \partial W$ and $f: M \to N$ extends to all of W, then $\deg_{\mathbb{Z}/2}(f) = 0$.

<u>Ex</u> A constant map $c: M \to M$, $x \mapsto c$ has $\deg_{\mathbb{Z}/2}(c) = 0$.

The identity map $id: M \to M$, $x \mapsto x$ has $\deg_{\mathbb{Z}/2}(id) = 1$.

In particular, the identity map of a compact manifold (without boundary) is NOT homotopic to a constant map.
(In case $M = \partial W$, this implies that $id: M \to M$ cannot be extended to all of W, a result we saw in Lecture 10.)